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Gravitational and Electroweak Unification

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Einstein suggested that a unified field theory be constructed by replacing the diffeomorphisms (the coordinate transformations of general relativity) with some larger group. We have constructed a theory that unifies the gravitational and electroweak fields by replacing the diffeomorphisms with the largest group of coordinate transformations under which conservation laws are covariant statements. This replacement leads to a theory with field equations which imply the validity of the Einstein equations of general relativity, with a stress-energy tensor that is just what one expects for the electroweak field and associated currents. The electroweak field appears as a consequence of the field equations (rather than as a "compensating field" introduced to secure gauge invariance). There is no need for symmetry breaking to accommodate mass, because the $U(1) \times SU(2)$ gauge symmetry is approximate from the outset. The gravitational field is described by the space-time metric, as in general relativity. The electroweak field is described by the "mixed symmetry" part of the Ricci rotation coefficients. The gauge symmetry-breaking quantity is a vector formed by contracting the Levi-Civita symbol with the totally antisymmetric part of the Ricci rotation coefficients.

1. INTRODUCTION

In his autographical notes, Einstein (1949) suggested that the construction of a unified field theory "would be most beautiful, if one were to succeed in expanding the group once more, analogous to the step which led from special relativity to general relativity." This suggestion was in accord with the prophetic remark by Dirac (1930) that "Further progress lies in the direction of making our equations invariant under wider and still wider transformations." For several decades, we have been engaged in a program (Pandres, 1962, 1981, 1984a, b, 1995, 1998) in which we have pursued Einstein's suggestion that the diffeomorphisms (the covariance group for general relativity) somehow be extended to a larger group. This program has

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now led to a theory which unifies the gravitational field with a field that appears to describe the electroweak field. There is reason to believe that the quantized theory may be finite or renormalizable, and free of anomalies. For the purpose of optimizing readability, we devote this Introduction to a brief overview, without proofs, of the main results from prior papers and of several new results. In subsequent sections, we present detailed developments with proofs of these results. We make minor changes in notation and terminology to enhance clarity.

A diffeomorphism from space-time coordinates x^{α} to space-time coordinates $x^{\tilde{\alpha}}$ satisfies the commutation condition

$$[\partial_{\mu}, \partial_{\nu}]x^{\tilde{\alpha}} = 0 \tag{1}$$

where $[\partial_{\mu}, \partial_{\nu}] = \partial_{\mu}\partial_{\nu} - \partial_{\nu}\partial_{\mu}$, and ∂_{μ} denotes partial differentiation with respect to x^{μ}. (Partial differentiation is denoted also by a comma, e.g., $\partial_{\mu}x^{\tilde{\alpha}} = x^{\tilde{\alpha}}_{,\mu}$).

Initially, we merely suggested (Pandres, 1962) that equation (1) be discarded. Our suggestion was motivated by an argument which is a generalization of the "elevator" argument that led Einstein from special relativity to general relativity. Our argument provided reason to believe that discarding equation (1) might lead to unification of the gravitational and electroweak fields. In Section 2, we recall this argument, and present a new result that makes the argument far more compelling. New results are also included in Sections 4 and 5.

Subsequently, we suggested more specifically (Pandres, 1981) that equation (1) be replaced by the weaker condition

$$x^{\nu}_{,\tilde{\alpha}}\,[\partial_{\mu}, \partial_{\nu}]x^{\tilde{\alpha}} = 0 \tag{2}$$

Coordinate transformations that satisfy equation (2) are called *conservative*. In Section 3.3, we recall that these transformations form a group, which we call the *conservation group* because it is the largest group of coordinate transformations under which conservation laws are covariant statements. It contains the diffeomorphisms as a proper subgroup. Our theory is detemined by the conservation group—as general relativity is determined by the diffeomorphisms, as special relativity is determined by the Lorentz group, and as Newtonian theory is determined by the Galilei group.

In Section 3.1, we recall in detail why commutation of partial derivatives is not to be taken for granted. Briefly, the reason is that partial derivatives are defined on a class of *functionals on paths* $F(p)$ that contains the ordinary functions $F(x)$ as a subclass. The new coordinates $x^{\tilde{\alpha}}$ are such path-dependent functionals. Thus, our theory is based, not on a Riemannian manifold, but rather on a space in which paths (directed curve segments) are the most primitive entities that have invariant meaning. The properties of our path space are recalled in Section 3.2.

Discussions of physical theories based on a Riemannian manifold often use the index-free notation of differential forms. This notation exhibits the geometry of a Riemannian manifold and the integrability conditions of equations defined on such a manifold in an especially transparent and suggestive way. However, because our theory is based on a space which is more general than a Riemannian manifold, we refrain from using the notation of forms. [When we abandon equation (1), we abandon the Poincaré lemma $\mathbf{dd} = \mathbf{0}$, which expresses (see, e.g., De Felice and Clarke, 1990) the equality of mixed partial derivatives with respect to space-time coordinates. Also, the forms notation is most convenient for the discussion of vectors and of rank-two *antisymmetric* tensors. It is less convenient for the discussion of rank-two *symmetric* tensors, and actually obscures the discussion of *rank-three tensors* (which may be decomposed into their symmetric, antisymmetric, and mixed-symmetry parts). Such rank-three tensors will play a crucial role in the development of our theory.] We use tensor notation and the summation convention. Greek and Latin indices take the values 0, 1, 2, 3.

We also recall the geometry which is determined on path space by the conservation group (as Riemannian geometry is determined on a manifold by the diffeomorphisms). The structure of this geometry is expressed by a tetrad of vectors h^i_μ. The tetrad is the primary quantity in our theory. The metric is a secondary quantity, defined in terms of the tetrad by $g_{\mu\nu} = g_{ij} h^i_\mu h^j_\nu$, where $g_{ij} = g^{ij} = \text{diag}(-1, 1, 1, 1)$. Latin (tetrad) indices are raised and lowered using g^{ij} and g_{ij}, just as Greek (space-time) indices are raised and lowered using $g^{\mu\nu}$ and $g_{\mu\nu}$. We define a quantity

$$C_\mu = h^\nu_i \, (h^i_{\mu,\nu} - h^i_{\nu,\mu}) \tag{3}$$

which is a vector under the conservation group. We call this the *curvature vector* because there exists a conservative coordinate transformation to a coordinate system in which the tetrad is constant if and only if C_μ vanishes.

The quantity $C^\mu C_\mu$ is invariant under the conservation group. In Section 4, we show that this invariant is an appropriate Lagrangian for gravitational and electroweak unification. In Section 5, we consider the field equations that flow from the variational principle

$$\delta \int C^\mu C_\mu \, \sqrt{-g} \, d^4x = 0 \tag{4}$$

where the 16 components of h^i_μ are varied independently. In Section 5.2, we show that the set of tetrads h^i_α which satisfy these field equations contains a nondenumerably infinite proper subset of path-independent tetrads, i.e., tetrads for which the condition

$$[\partial_\mu, \partial_\nu] h^i_\alpha = 0 \tag{5}$$

is satisfied. This condition is covariant under diffeomorphisms. Thus, any h^i_α in this subset appears in the guise of a tetrad defined on a Riemannian manifold. In the most fundamental sense, however, our geometry remains based not on a Riemannian manifold, but on path space.

The Einstein equations of general relativity may be interpreted in two ways. One interpretation is as differential equations for the metric when the stress-energy tensor is given. Alternatively, these equations may be looked upon as a definition of the stress-energy tensor in terms of the metric. The second interpretation has been stressed particularly by Schrödinger (1960) ["I would rather you did not regard these equations as field equations, but as a definition of T_{ik} the matter tensor"] and by Eddington (1924) ["and we must proceed by inquiring first what experimental properties the physical tensor possesses, and then seeking a geometrical tensor which possesses these properties"]. It is the second interpretation that we adopt. In Section 5.3, we show that our field equations restrict the metric in such a way as to imply the validity of the Einstein equations with a stress-energy tensor that is just what one expects for the electroweak field and associated currents. In Section 6, we discuss a different variational principle that yields different field equations which, however, imply the validity of Einstein equations that are identical in form to those of Section 5.3, but slightly different in interpretation.

2. MOTIVATION

In our first paper on field unification (Pandres, 1962) we began with the special relativistic equation of motion for a free particle

$$\frac{d^2 x^i}{ds^2} = 0 \tag{6}$$

where $-ds^2 = g_{ij} dx^i dx^j$. We considered the image equation of this free-particle equation under a transformation from coordinates x^i to coordinates x^α, where $[\partial_\mu, \partial_\nu] x^i$ is not zero. We denote $x^i_{,\mu}$ by h^i_μ, so we see that the curl $f^i_{\mu\nu} = h^i_{\nu,\mu} - h^i_{\nu,\mu}$ is not zero. From the chain rule for differentiation, we have $dx^i/ds = h^i_\mu \, dx^\mu/ds$. Upon differentiating this with respect to s, using the chain rule, and multiplying by h^α_i, we see that equation (6) may be written

$$\frac{d^2 x^\alpha}{ds^2} + h^\alpha_i h^i_{\mu,\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0 \tag{7}$$

We follow Eisenhart (1925) in defining Ricci rotation coefficients by $\gamma^i{}_{\mu\nu} = h^i_{\mu;\nu} = h^i_{\mu,\nu} - h^i_\sigma \Gamma^\sigma_{\mu\nu}$, where a semicolon denotes the usual covariant differentiation with respect to the Christoffel symbol $\Gamma^\alpha_{\mu\nu}$. Multiplication by h^α_i gives $h^\alpha_i h^i_{\mu,\nu} = \Gamma^\alpha_{\mu\nu} + \gamma^\alpha{}_{\mu\nu}$, and upon using this in equation (7) we have

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = -\gamma^\alpha{}_{\mu\nu} \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} \tag{8}$$

(Note: The relation $\gamma_{\mu\nu i} = h^j_\mu \, \gamma_{j\nu\alpha} h^\alpha_i$ illustrates our general method for converting between Greek and Latin indices.)

Now, the affine connection for spin in general relativity is expressed in terms of the Ricci rotation coefficients by $\Gamma_\mu = 1/8 \, \gamma_{ij\mu}(\gamma^i\gamma^j - \gamma^j\gamma^i) + a_\mu I$, where the γ^i are the Dirac matrices of special relativity, I is the identity matrix, and a_μ is an arbitrary vector. It is well known that the spin connection contains complete information about the electromagnetic field, and that one-half of Maxwell's equations are identically satisfied on account of the existence of the spin connection. Furthermore, the manner in which the electromagnetic field enters the spin connection is in agreement with the principle of minimal electromagnetic coupling. An understanding of the spinor calculus in Riemann space, and the role played by the spin connection, was gained through the work of many investigators during the decade after Dirac's discovery of the relativistic theory of the electron; see, e.g., Bade and Jehle (1953) for a general review. Many of these investigators recognized the description of the electromagnetic field as part of the spin connection. An especially lucid discussion of this was given by Loos (1963). The subsequent unification of the electromagnetic and weak fields by Weinberg (1967) and Salam (1968) leads us to expect that the spin connection might also contain a description of the weak field.

We now recall (Pandres, 1995) evidence that the electroweak field is described by $M_{\mu\nu i}$, the "mixed symmetry" part of $\gamma_{\mu\nu i}$ under the permutation group on three symbols. The totally symmetric part vanishes because $\gamma_{\mu\nu i}$ is antisymmetric in μ and ν. Thus, we have $\gamma_{\mu\nu i} = M_{\mu\nu i} + A_{\mu\nu i}$ where $A_{\mu\nu i}$ is the totally antisymmetric part. Clearly, $A^\alpha_{\mu\nu}$ makes no contribution to the right side of equation (8), so

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = \frac{dx^\mu}{ds} M_\mu{}^\alpha{}_i v^i \tag{9}$$

where $v^i = dx^i/ds$ is the (constant) first integral of equation (6). The totally antisymmetric part of $\gamma_{\mu\nu i}$ is

$$A_{\mu\nu i} = \frac{1}{3}(\gamma_{\mu\nu i} + \gamma_{i\mu\nu} + \gamma_{\nu i\mu}) \tag{10}$$

Thus, the mixed symmetry part is $M_{\mu\nu i} = \gamma_{\mu\nu i} - A_{\mu\nu i}$, so, we have

$$M_{\mu\nu i} = \frac{1}{3}(2\gamma_{\mu\nu i} - \gamma_{i\mu\nu} - \gamma_{\nu i\mu}) \tag{11}$$

The antisymmetry of $\gamma_{\mu\nu i}$ in its first two indices may be used to obtain an expression for $M_{\mu\nu i}$ in terms of $f_{i\mu\nu}$. We have $f_{i\mu\nu} = h_{i\nu,\mu} - h_{i\mu,\nu} = h_{i\nu;\mu} - h_{i\mu;\nu}$, so that $f_{i\mu\nu} = \gamma_{i\nu\mu} - \gamma_{i\mu\nu}$. If we subtract from this the corresponding expressions for $f_{\mu\nu i}$ and $f_{\nu i\mu}$, we see that $\gamma_{\mu\nu i} = 1/2\ (f_{i\mu\nu} - f_{\mu\nu i} - f_{\nu i\mu})$. By using this and the corresponding expressions for $\gamma_{i\mu\nu}$ and $\gamma_{\nu i\mu}$ in equation (11), we obtain

$$M_{\mu\nu i} = \frac{1}{3}(2f_{i\mu\nu} - f_{\mu\nu i} - f_{\nu i\mu}) \tag{12}$$

which may be written

$$M_{\mu\nu i} = \frac{1}{3}(2\delta_i^n\delta_\mu^\alpha\delta_\nu^\sigma - h_\mu^n\delta_\nu^\alpha\, h_i^\sigma - h_\mu^n\, h_i^\alpha\, \delta_\mu^\sigma)\, f_{n\alpha\sigma} \tag{13}$$

where δ_μ^α is the Kronecker delta. It is important to notice that equation (13) may be rewritten in the form

$$M_{\mu\nu i} = \frac{1}{3}(2\delta_i^n\, \delta_\mu^\alpha\delta_\nu^\sigma - h_\mu^n\, \delta_\nu^\alpha\, h_i^\sigma - h_\nu^n\, h_i^\alpha\, \delta_\mu^\sigma)\mathfrak{F}_{n\alpha\sigma} \tag{14}$$

where

$$\mathfrak{F}_{i\mu\nu} = f_{i\mu\nu} + e_{0ij\kappa}h^j_\mu\, h^k_\nu \tag{15}$$

and $e_{nij\kappa}$ is the Levi-Civita symbol. In rewriting equation (13) as equation (14), we used the easily verifiable fact that

$$(2\delta_i^n\, \delta_\mu^\alpha\, \delta_\nu^\sigma - h_\mu^n\, \delta_\nu^\alpha\, h_i^\sigma - h_\nu^n\, h_i^\alpha\, \delta_\mu^\sigma)\, e_{0njk}\, h^j_\alpha\, h^k_\sigma = 0$$

Now, $\mathfrak{F}_{i\mu\nu}$ is the usual field strength (see, e.g., Nakahara, 1990) for a $U(1) \times SU(2)$ gauge field, *provided that* h^i_μ *is transformed on its tetrad indices as a gauge potential*, rather than as a Lorentz vector. If h^i_μ is transformed as a gauge potential, the metric $g_{\mu\nu} = g_{ij}h^i_\mu h^j_\nu$ is generally changed.

It is eminently reasonable that when a particle is subjected to a gauge transformation which changes its mass, the gravitational field should change.

From equation (14), we see that in expression (13), for $M_{\mu\nu i}$ the curl $f_{n\alpha\sigma}$ may simply be replaced by the gauge field $\mathfrak{F}_{n\alpha\sigma}$. We shall see that the quantity $\mathfrak{F}_{i\mu\nu}$ does not directly describe the electroweak field. It is, however, the fundamental ingredient which is essential for the description of that field. The $\mathfrak{F}_{n\alpha\sigma}$ in equation (14) may be viewed as a field with "bare" or massless quanta which are "clothed" by the factor $1/3\ (2\delta_i^n\delta_\mu^\alpha\delta_\nu^\sigma - h_\mu^n\delta_\nu^\alpha h_i^\sigma -$

$h^n_\nu h^\alpha_i \delta^\sigma_\mu$) and thus may acquire mass. It is $M_{\mu\nu i}$ that we identify as the physical electroweak field, and which (as we shall see in Sections 4 and 5) appears in the appropriate way in our Lagrangian and in the stress-energy tensor of the Einstein equations. For this identification to be valid, the quantity $M_{\mu\nu 0} = 1/3\ (2f_{0\mu\nu} - f_{\mu\nu 0} - f_{\nu 0\mu})$ must describe the electromagnetic field; hence, it must be the curl of a vector. The presence of the terms $-f_{\mu\nu 0} - f_{\nu 0\mu}$ may cause one to ask how $M_{\mu\nu i}$ can be identified as the electroweak field. Our answer is this: The orthodox physical interpretation, which we adopt, is that h^i_μ describes an observer frame. Now, if h^i_μ *describes a freely falling, nonrotating observer frame, our expression for* $M_{\mu\nu 0}$ *reduces to* $M_{\mu\nu 0} = 1/3\ f_{0\mu\nu}$. This may be seen as follows. The condition for a freely falling, nonrotating frame (Synge, 1960) is $h_{i\nu;\alpha} h^\alpha_0 = 0$. In terms of the Ricci rotation coefficients, the condition is $\gamma_{\mu\nu 0} = 0$. From this and equation (11), we see that for an h^i_μ which describes a freely falling, nonrotating observer frame,

$$M_{\mu\nu 0} = \frac{1}{3}(\gamma_{0\nu\mu} - \gamma_{0\mu\nu}) = \frac{1}{3}(h_{0;\mu\nu} - h_{0\mu;\nu}) = \frac{1}{3}(h_{0\nu,\mu} - h_{0\mu,\nu}) = \frac{1}{3} f_{0\mu\nu}$$

Moreover, in the nonrelativistic limit (i.e., for v^1, v^2, v^2 small compared to one), the electromagnetic term $(dx^\mu/ds)\ M_\mu{}^\alpha{}_0 v^0$ dominates the right side of equation (9).

3. MATHEMATICAL PRELIMINARIES

Any ordered set of four independent real variables x^α may be regarded as coordinates of points in a four-dimensional arithmetic space X.

3.1. Path-Dependent Functionals

3.1.1. Paths

Let $x^\alpha(\lambda)$ be continuous functions of a real parameter λ on the interval $-\infty < \lambda < \infty$. By a *path p*, we mean the set of all points in X that are identified by $x^\alpha = x^\alpha(\lambda)$ for $-\infty < \lambda \leq \Lambda$. Thus, one endpoint of a path p is the point i with coordinates $\lim_{\lambda \to -\infty} x^\alpha(\lambda)$, while the other endpoint is the point x with coordinates $x^\alpha(\Lambda)$. We regard i as the initial point, and x as the terminus, of p. The set of all paths p is regarded as a space of paths and is denoted by P.

3.1.2. Path-Dependent Functionals and Their Derivatives

Let F be a path-dependent functional, i.e., a rule that assigns to each path p a real number $F(p)$. Following the method introduced by us (Pandres,

1962) and independently by Mandelstam (1962), we define derivatives of $F(p)$ by giving p an extension from its terminus x, while holding the rest of p completely fixed. Any path may be extended in this way by extending the domain of $x^{\alpha}(\lambda)$ to the interval $-\infty < \lambda \leq \Lambda + \Delta\Lambda$, where $\Delta\Lambda > 0$. The resulting path $p + \Delta p$ is called a path extended from p, and the set of all points in X that are defined by $x^{\alpha} = x^{\alpha}(\lambda)$ for $\Lambda < \lambda \leq \Lambda + \Delta\Lambda$ is called an extension of p and is denoted by Δp. If, for each path p and each extension Δp, the condition $\lim_{\Delta\Lambda\to 0} [F(p + \Delta p) - F(p)] = 0$ is satisfied, we call F a *normal* functional. We limit our considerations to normal functionals. We define F' by

$$F' = \lim_{\Delta\Lambda\to 0} \frac{F(p + \Delta p) - F(p)}{\Delta\Lambda}$$

If the extension Δp is chosen so that, along it, only a single coordinate x^{β} changes, and if the parametrization is such that on this extension $\Delta\Lambda = \Delta x^{\beta}$, then F' is called the partial derivative of F with respect to x^{β}, and denoted by $\partial_{\beta}F$ or by $F_{,\beta}$. If, along Δp, the coordinate increments Δx^{β} are unrestricted and independent, then F' is called the total derivative of F with respect to Λ, and is denoted by $dF/d\Lambda$. It is also convenient to denote $dx^{\alpha}/d\lambda$, evaluated for $\lambda = \Lambda$, by $dx^{\alpha}/d\Lambda$. If the partial derivatives and the total derivative of F are related in such a way that the chain rule for differentiation is valid, i.e., if $dF/d\Lambda = F_{,\alpha}\, dx^{\alpha}/d\Lambda$, then F is called a *smooth* functional. A smooth functional whose partial derivatives of all orders are also smooth is called a *regular* functional. We limit our considerations to regular functionals. When we wish to emphasize the path-dependent character of a functional F, we use the notation $F(p)$. Our functionals include, as a subclass, the ordinary functions of x, i.e., functionals which are "path-dependent" in the trivial sense that they depend only on the terminus x of a path p; for them, we use the notation $F(x)$.

3.1.3. Noncommutativity of Partial Derivatives

From the path p, let two extended paths $p + \Delta p_1$ and $p + \Delta p_2$ be constructed such that the extensions Δp_1 and Δp_2 do not coincide, but such that the termini of $p + \Delta p_1$ and $p + \Delta p_2$ *do* coincide. The values of $F(p + \Delta p_1)$ and $F(p + \Delta p_2)$ are not generally equal. By letting Δp_1 be an extension along which first only x^{ν} changes and then only x^{μ} changes, and letting Δp_2 be an extension along which first only x^{μ} changes and then only x^{ν} changes, we see that $\partial_{\mu}\partial_{\nu}F$ equals $\partial_{\nu}\partial_{\mu}F$ for functions $F(x)$, but not generally for functionals $F(p)$.

3.2. Path Space

3.2.1. *The Requirement That No Preferred Coordinate System Shall Exist*

From the chain rule, we have $F_{,\nu} = F_{,\tilde{\sigma}} x^{\tilde{\sigma}}_{,\nu}$. If we differentiate with respect to x^{μ} and subtract the corresponding expression with μ and ν interchanged, we get $[\partial_{\mu},\partial_{\nu}]\, F = x^{\tilde{\rho}}_{,\mu} x^{\tilde{\sigma}}_{,\nu}\, [\partial_{\tilde{\rho}},\partial_{\tilde{\sigma}}]\, F + F_{,\tilde{\sigma}}\, [\partial_{\mu},\partial_{\nu}]\, x^{\tilde{\sigma}}$. Consider a transformation for which $[\partial_{\mu},\partial_{\nu}]\, x^{\sigma}$ does not vanish. If we were to demand that $[\partial_{\mu},\partial_{\nu}]$ F vanish, then we would find that $[\partial_{\tilde{\rho}},\partial_{\tilde{\sigma}}]\, F$ does not generally vanish. Thus, the coordinates x^{α} and $x^{\tilde{\alpha}}$ would not be on an equal footing; i.e., the coordinates x^{α} would be "preferred." The requirement that x^{α} and $x^{\tilde{\alpha}}$ be on an equal footing compels us to consider a space in which paths, rather than points, are the primary elements.

3.2.2. *Abstract Path Space*

Just as the x^{α} are regarded as coordinates of points x in the arithmetic space X and the set of all paths p is regarded as a space of paths P, *another* ordered set of four independent real variables $x^{\tilde{\alpha}}$ may be regarded as coordinates of points $\tilde{x}$ in *another* four-dimensional arithmetic space $\tilde{X}$, and the set of all paths $\tilde{p}$ may be regarded as *another* space of paths $\tilde{P}$. Let M be a one-to-one mapping from P onto $\tilde{P}$; let $\tilde{p}$ be the image path of p, and let $\tilde{x}$ be the terminus of $\tilde{p}$. Since $\tilde{x}$ is determined by $\tilde{p}$, and $\tilde{p}$ is determined by p (via the mapping M), it is clear that the coordinates $x^{\tilde{\alpha}}$ are functionals of p, i.e., $x^{\tilde{\alpha}} = x^{\tilde{\alpha}}(p)$. Similarly, $x^{\alpha} = x^{\alpha}(\tilde{p})$. If the image path of each path extended from p is a path extended from $\tilde{p}$, and if $x^{\tilde{\alpha}}(p)$ and $x^{\alpha}(\tilde{p})$ are regular functionals, then M is called a regular mapping. We limit our considerations to regular mappings.

We began by regarding a mapping M as a *path transformation* (which maps each path p in P to a path $\tilde{p}$ in $\tilde{P}$ and conversely). There is, however, another point of view that is more interesting and useful, and that we now adopt: We introduce an *abstract path space* Π in which (abstract) paths π are the primary elements, and regard M as a path-dependent coordinate transformation $x^{\tilde{\alpha}} = x^{\tilde{\alpha}}(p)$ that merely changes the arithmetic-space framework for discussing Π. The arithmetic spaces X and $\tilde{X}$ provide equivalent frameworks for discussing Π, and the path spaces P and $\tilde{P}$ are equivalent representations of Π. A path p and its image path $\tilde{p}$ are equivalent representations of the same abstract path π in Π. The changed point of view that we have adopted is analogous to that in which one begins by regarding a suitable transformation $x^{\tilde{\alpha}} = x^{\tilde{\alpha}}(x)$ as a mapping from a point x to a point $\tilde{x}$, and then recognizes that it is more interesting and useful to regard the transformation as a diffeomorphism, in which the same point of an abstract point space (a manifold) is merely relabeled with new coordinate values. The coordinates

x^{α} and $x^{\tilde{\alpha}}$ provide equivalent coordinate systems for discussing Π, but the points x and $\tilde{x}$ that x^{α} and $x^{\tilde{\alpha}}$ identify in X and $\tilde{X}$, respectively, have no meaning in Π. This is clear, because a path-dependent coordinate transformation does not generally establish a one-to-one correspondence between points of X and $\tilde{X}$, even in coordinate patches. Two paths which have the same termini in X have image paths with different termini in $\tilde{X}$, and conversely. The correspondence between x and $\tilde{x}$ is both one-to-many and many-to-one (hence, nonunique in both directions).

We suggest that physical space is described by our path space Π. Skepticism that a Riemannian manifold adequately describes physical space has been expressed by many investigators (e.g., Eddington, 1924; Penrose, 1968; Penrose and MacCallum, 1973; Finkelstein, 1969, 1972a,b, 1974; Finkelstein *et al.*, 1974; Bergmann and Komar, 1972; Gambrini and Trias, 1981). The following comments of Eddington (1924) most nearly anticipate our approach:

> There is a certain hiatus in the arguments of the relativity theory which has never been thoroughly explored. . . . the arbitrariness of the coordinate-system is limited. We may apply any continuous transformation; but our theory does not contemplate a discontinuous transformation of coordinates, such as would correspond to a reshuffling of the points of the continuum. There is something corresponding to an *order of enumeration* of the points which we desire to preserve, when we limit the changes of coordinates to continuous transformations. . . . The hiatus probably indicates something more than a temporary weakness of the rigorous deduction. It means that space and time are only approximate conceptions, which must ultimately give way to a more general conception of the ordering of events in nature . . . .

3.3. The Conservation Group and the Curvature Vector

A relativistic conservation law is an expression of the form $V^{\alpha}_{,\alpha} = 0$, where V^{α} is a vector density of weight $+1$. This is a covariant statement under a path-dependent coordinate transformation relating x^{α} and $x^{\tilde{\alpha}}$ if and only if it implies and is implied by the relation $V^{\tilde{\alpha}}_{,\tilde{\alpha}} = 0$. The transformation law for a vector density of weight $+1$ is $V^{\tilde{\alpha}} = (\partial x/\tilde{x})\, x^{\tilde{\alpha}}{}_{,\mu}\, V^{\mu}$, where $\partial x/\tilde{x}$ is the (nonzero) Jacobian determinant of $x^{\mu}_{,\tilde{\alpha}}$. Upon differentiating $V^{\tilde{\alpha}}$ with respect to $x^{\tilde{\alpha}}$, we obtain

$$V^{\tilde{\alpha}}_{,\tilde{\alpha}} = \left(\frac{\partial x}{\partial \tilde{x}}\, x^{\tilde{\alpha}}_{,\mu} \right)_{,\tilde{\alpha}} V^{\mu} + \frac{\partial x}{\partial \tilde{x}}\, V^{\alpha}_{,\alpha}$$

For arbitrary V^{μ}, we see that a conservation law is a covariant statement if and only if

$$\left(\frac{\partial x}{\partial \tilde{x}} x^{\tilde{\alpha}}_{,\mu}\right)_{,\tilde{\alpha}} = 0 \tag{16}$$

For this reason, we call a path-dependent coordinate transformation *conservative* if it satisfies equation (16). Now,

$$\left(\frac{\partial x}{\partial \tilde{x}} x^{\tilde{\alpha}}_{,\mu}\right)_{,\tilde{\alpha}} = \left(\frac{\partial x}{\partial \tilde{x}}\right)_{,\alpha} x^{\tilde{\alpha}}_{,\mu} + \frac{\partial x}{\partial \tilde{x}} x^{\tilde{\alpha}}_{,\mu,\tilde{\alpha}} = \partial_\mu \frac{\partial x}{\partial \tilde{x}} + \frac{\partial x}{\partial \tilde{x}} x^{\tilde{\alpha}}_{,\mu,\nu}\, x^{\nu}_{,\tilde{\alpha}}$$

so, if we use the well-known formula

$$\partial_\mu \frac{\partial x}{\partial \tilde{x}} = \frac{\partial x}{\partial \tilde{x}} x^{\tilde{\alpha}}_{,\nu} \partial_\mu x^{\nu}_{,\tilde{\alpha}}$$

for the derivative of a determinant, and note that $x^{\tilde{\alpha}}_{,\nu} x^{\nu}{}_{,\tilde{\alpha},\mu} = -\, x^{\tilde{\alpha}}_{,\nu,\mu} x^{\nu}{}_{,\tilde{\alpha}}$, we find that equation (16) is equivalent to our equation (2), i.e.,

$$x^{\nu}_{,\tilde{\alpha}}\, [\partial_\mu, \partial_\nu]\, x^{\tilde{\alpha}} = 0$$

3.3.1. The Conservation Group

We now recall (Pandres, 1981) an explicit proof that the conservative coordinate transformations form a group. [Finkelstein (1981), however, pointed out that the group property follows implicitly from the derivation given above.] First, we note that the identity transformation $x^{\tilde{\alpha}} = x^{\alpha}$ is a conservative coordinate transformation. Next, we consider the result of following a coordinate transformation from x^{α} to $x^{\tilde{\alpha}}$ by a coordinate transformation from $x^{\tilde{\alpha}}$ to $x^{\hat{\alpha}}$. Upon differentiating

$$x^{\hat{\alpha}}_{,\mu} = x^{\hat{\alpha}}_{,\tilde{\rho}} x^{\tilde{\rho}}_{,\mu} \tag{17}$$

with respect to x^{ν}, subtracting the corresponding expression with μ and ν interchanged, and multiplying by $x^{\nu}_{,\hat{\alpha}}$, we obtain

$$x^{\nu}_{,\hat{\alpha}}\, [\partial_\mu, \partial_\nu]\, x^{\hat{\alpha}} = x^{\tilde{\rho}}_{,\mu} x^{\tilde{\sigma}}_{,\hat{\alpha}}\, [\partial_{\tilde{\rho}}, \partial_{\tilde{\sigma}}]\, x^{\hat{\alpha}} + x^{\nu}_{,\tilde{\rho}}\, [\partial_\mu, \partial_\nu]\, x^{\tilde{\rho}} \tag{18}$$

We see from equation (18) that if $x^{\nu}_{,\tilde{\rho}}\, [\partial_\mu, \partial_\nu]\, x^{\tilde{\rho}}$ and $x^{\tilde{\sigma}}_{,\hat{\alpha}}\, [\partial_{\tilde{\rho}}, \partial_{\tilde{\sigma}}]\, x^{\hat{\alpha}}$ vanish, then $x^{\nu}_{,\hat{\alpha}}\, [\partial_\mu, \partial_\nu]\, x^{\hat{\alpha}}$ vanishes. This shows that if the transformations from x^{α} to $x^{\tilde{\alpha}}$ and from $x^{\tilde{\alpha}}$ to $x^{\hat{\alpha}}$ are conservative coordinate transformations, then the product transformation from x^{α} to $x^{\hat{\alpha}}$ is a conservative coordinate transformation. If we let $x^{\hat{\alpha}} = x^{\alpha}$, we see from equation (18) that the inverse of a conservative coordinate transformation is a conservative coordinate transformation. From equation (17), we see that the product of matrices $x^{\tilde{\rho}}_{,\mu}$ and $x^{\hat{\alpha}}_{,\tilde{\rho}}$ (which represent the transformations from x^{α} to $x^{\tilde{\alpha}}$ and from $x^{\tilde{\alpha}}$ to $x^{\hat{\alpha}}$, respectively) equals the matrix $x^{\hat{\alpha}}_{,\mu}$ (which represents the product transformation from x^{α} to $x^{\hat{\alpha}}$). It is obvious, and well known, that if products admit a matrix representation in this sense, then the associative law is satisfied. This completes the proof that the conservative coordinate transformations form a group, which we call the conservation group.

To show that the conservation group contains the diffeomorphisms as a proper subgroup, we need only exhibit a coordinate transformation that satisfies equation (2), but does not satisfy equation (1). Such a coordinate transformation is

$$x^{\tilde{\alpha}} = x^{\alpha} + \delta^{\alpha}_{0} \int_{i}^{x} x^{1} dx^{2} \tag{19}$$

where the integral from i to x is taken along the path p. We see by inspection that the inverse of the transformation defined in equation (19) is

$$x^{\nu} = x^{\tilde{\nu}} - \delta^{\nu}_{0} \int_{\tilde{i}}^{\tilde{x}} x^{\tilde{1}} dx^{\tilde{2}} \tag{20}$$

where the integral from $\tilde{i}$ to $\tilde{x}$ is taken along the path $\tilde{p}$. Upon differentiating equation (19) with respect to x^{ν}, we obtain $x^{\tilde{\alpha}}_{,\nu} = \delta^{\alpha}_{\nu} + \delta^{\alpha}_{0}\delta^{2}_{\nu}\, x^{1}$. By differentiating this with respect to x^{μ} and subtracting the corresponding expression with μ and ν interchanged, we obtain

$$[\partial_{\mu}, \partial_{\nu}]\, x^{\tilde{\alpha}} = \delta^{\alpha}_{0}\, (\delta^{1}_{\mu}\, \delta^{2}_{\nu} - \delta^{1}_{\nu}\delta^{2}_{\mu}) \tag{21}$$

A nonzero component of equation (21) is $[\partial_{1}, \partial_{2}]\, x^{\tilde{0}} = 1$, which shows that equation (1) is not satisfied. Upon differentiating equation (20) with respect to $x^{\tilde{\alpha}}$, we obtain $x^{\nu}_{,\tilde{\alpha}} = \delta^{\nu}_{\alpha} - \delta^{\nu}_{0}\delta^{2}_{\alpha}x^{\tilde{1}}$. If we multiply equation (21) by this, we see that equation (2) is satisfied.

3.3.2. The Curvature Vector

The geometry determined on a manifold by the diffeomorphisms is Riemannian geometry, whose structure is expressed by a symmetric metric $g_{\mu\nu}$. From $g_{\mu\nu}$ and its derivatives, one defines an object $R^{\alpha}{}_{\beta\mu\nu}$ which is a tensor under the diffeomorphisms, and which is called the Riemann tensor. There exists a diffeomorphism from x^{α} to a coordinate system in which the metric is constant, if and only if $R^{\alpha}{}_{\beta\mu\nu}$ vanishes.

We now consider the geometry which is determined on path space, in an analogous way, by the conservation group. [We would say "determined by the conservation group in the sense of Klein's (1893) Erlanger Program," but this could cause some confusion. Roughly speaking, Klein's program states that a group of transformations on a space determines a geometry on the space, and conversely; however, mathematicians appear to differ somewhat concerning the precise modern interpretation of Klein's program. See, e.g, Weyl (1931) and Millman (1977).]

The structure of path space is expressed by a tetrad h^i_μ. [Note: Tetrads were used in physics by Einstein (1928a,B) under the name "vierbein." The mathematical properties of tetrads were made available to physicists soon afterward in two lucid papers by Weitzenböck (1928) and Levi-Civita (1929). A large literature has since developed. For example, Rosenfeld (1930) suggested that tetrads present certain advantages for the formulation of a quantum field theory of gravitation (in the very first paper on that subject). Utiyama (1956) and Kibble (1961) used tetrads in conjunction with attempts at understanding gravitation as a compensating field in the sense of Yang and Mills (1954). The Petrov (1969) classification of the Weyl tensor was originally developed using tetrads (though a development using spinors is easier to understand). Möller (1961) used tetrads in a revival of Einstein's theory of distant parallelism. A nice summary of the tetrad literature has been given by De Felice and Clarke (1990).]

Under a coordinate transformation, the tetrad transforms as a vector, i.e.,

$$h^i_\mu = h^{\tilde{i}}_{\tilde{\alpha}}\, x^{\tilde{\alpha}}_{,\mu} \tag{22}$$

We define an object C_μ, which we call the curvature vector, by

$$C_\mu = h^\nu_i \,(h^{\ i}_{\mu,\nu} - h^i_{\nu,\mu}) \tag{23}$$

Now, equation (23) may be written $C_\mu = -h^{i\nu} f_{i\mu\nu}$. By using equation (15), we verify that this may be rewritten into the form $C_\mu = -h^{i\nu}\mathfrak{F}_{i\mu\nu}$. It is important to notice that in the expression for C_μ, just as in the expression for $M_{\mu\nu i}$, the curl $f_{i\mu\nu}$ may simply be replaced by the gauge field $\mathfrak{F}_{i\mu\nu}$.

Upon differentiating equation (22) with respect to x^ν, subtracting the corresponding expression with μ and ν interchanged, and multiplying by h^ν_i we obtain

$$C_\mu = C_{\tilde{\alpha}} x^{\tilde{\alpha}}_{,\mu} - x^\nu_{,\tilde{\alpha}}\, [\partial_\mu, \partial_\nu]\, x^{\tilde{\alpha}} \tag{24}$$

where $C_{\tilde{\mu}} = h^{\tilde{\nu}}_i\,(h^{\tilde{i}}_{\tilde{\mu},\tilde{\nu}} - h^{\tilde{i}}_{\tilde{\nu},\tilde{\mu}})$. Equation (24) shows that C_μ is a vector under the conservation group. Now, *any* two tetrads h^i_μ and $h^{\tilde{i}}_{\tilde{\mu}}$ are related by a path-dependent coordinate transformation (not necessarily conservative). The relation is $x^{\tilde{\alpha}}_{,\mu} = h^{\tilde{\alpha}}_{\tilde{i}}\, h^i_\mu$. If $h^{\tilde{i}}_{\tilde{\mu}}$ is constant, then $C_{\tilde{\mu}}$ vanishes. Thus, we see from equation (24) that *there exists a conservative coordinate transformation to a coordinate system in which the tetrad is constant, if and only if the curvature vector vanishes.*

Upon multiplying equation (24) by h^μ_i, we obtain

$$C_i = \tilde{C}_i - h^\mu_i\, x^\nu_{\tilde{\alpha}}\, [\partial_\mu, \partial_\nu]\, x^{\tilde{\alpha}}$$

where $C_i = C_\mu\, h^\mu_i$ and $\tilde{C}_i = C_{\tilde{\mu}} h^{\tilde{\mu}}_i$. It is clear from this that *two tetrads are* *related by a conservative coordinate transformation if and only if they yield curvature vectors whose Latin components are identical.*

4. THE LAGRANGIAN

The quantity $C^{\mu}C_{\mu}$ is the only invariant that can be formed from the curvature vector by contraction with the metric. We now present evidence that this invariant is an appropriate Lagrangian for gravitational and electroweak unification.

The Riemann tensor is defined in the usual way by

$$R^{\alpha}{}_{\beta\mu\nu} = h_i^{\alpha}(h^i_{\beta;\mu;\nu} - h^i_{\beta;\nu;\mu})$$

while the Ricci tensor $R_{\mu\nu}$ and Ricci scalar R are defined, as usual, by $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$ and $R = R^{\alpha}{}_{\alpha}$. By using

$$h_i^{\alpha}\, h^i_{\beta;\mu;\nu} = (h_i^{\alpha}\, h^i_{\beta;\mu})_{;\nu} - h^{\alpha}_{i;\nu}\, h^i_{\beta;\mu} = \gamma^{\alpha}{}_{\beta\mu;\nu} + \gamma^{\alpha}{}_{\sigma\nu}\gamma^{\sigma}{}_{\beta\mu}$$

we easily find that

$$R^{\alpha}{}_{\beta\mu\nu} = \gamma^{\alpha}{}_{\beta\mu;\nu} - \gamma^{\alpha}{}_{\beta\nu;\mu} + \gamma^{\alpha}{}_{\sigma\nu}\gamma^{\sigma}{}_{\beta\mu} - \gamma^{\alpha}{}_{\sigma\mu}\gamma^{\sigma}{}_{\beta\nu} \tag{25}$$

From equation (23), we see that

$$C_{\mu} = h_i^{\nu}\,(h^i_{\mu,\nu} - h^i_{\nu,\mu}) = h_i^{\nu}\,(h^i_{\mu\nu} - h^i_{\nu;\mu}) = \gamma^{\nu}{}_{\mu\nu} - \gamma^{\nu}{}_{\nu\mu} = \gamma^{\nu}{}_{\mu\nu}$$

By using $C_{\mu} = \gamma^{\nu}_{\mu\nu}$, we find from equation (25) that

$$R_{\mu\nu} = C_{\mu;\nu} - C_{\alpha}\,\gamma^{\alpha}{}_{\mu\nu} - \gamma^{\alpha}{}_{\mu\nu;\alpha} + \gamma^{\alpha}{}_{\sigma\nu}\gamma^{\sigma}{}_{\mu\alpha} \tag{26}$$

and from equation (26)

$$C^{\mu}C_{\mu} = R + \gamma^{\mu i\nu}\,\gamma_{\mu\nu i} - 2C^{\mu}{}_{;\mu} \tag{27}$$

The first term on the right side of equation (27) is the Ricci scalar, which is the Lagrangian for gravitation. The last term is a covariant divergence, which contributes nothing to the field equations. We now consider the interpretation of the term $\gamma^{\mu i\nu}\,\gamma_{\mu\nu i}$. From equations (10) and (11), we see that

$$A^{\mu\nu i}M_{\mu\nu i} = 0 \tag{28}$$

and that

$$M_{\mu\nu i} + M_{i\mu\nu} + M_{\nu i\mu} = 0 \tag{29}$$

From $\gamma_{\mu\nu i} = M_{\mu\nu i} + A_{\mu\nu i}$ and equation (28), we get $\gamma^{\mu i\nu}\,\gamma_{\mu\nu i} = M^{\mu i\nu} M_{\mu\nu i} - A^{\mu\nu i}\,A_{\mu\nu i}$. But, $M^{\mu i\nu}M_{\mu\nu i} = 1/2\ M^{\mu i\nu}M_{\mu\nu i} + 1/2\ M^{\nu i\mu}M_{\nu\mu i} = 1/2\ M^{\mu i\nu}M_{\mu\nu i} + 1/2\ M^{i\nu\mu}M_{\mu\nu i} = 1/2\ (M^{\mu i\nu} + M^{i\nu\mu})\ M_{\mu\nu i} = 1/2\ M^{\mu\nu i}\ M_{\mu\nu i}$, where we have used equation (29). Thus, we have $\gamma^{\mu i \nu}\ \gamma_{\mu\nu i} = 1/2\ M^{\mu\nu i}\ M_{\mu\nu i} - A^{\mu\nu\alpha} A_{\mu\nu\alpha}$. We now define a vector

$$A^{\mu} = \frac{1}{3!} (-g)^{-1/2}\ e^{\mu\alpha\beta\sigma}\ A_{\alpha\beta\sigma} \tag{30}$$

and find that

$$A^{\mu\nu\alpha}\ A_{\mu\nu\alpha} = -6 A^{\mu}\ A_{\mu} \tag{31}$$

In obtaining equation (31), we have used the well-known identity (see, e.g., Weber, 1961) for expressing the product of two Levi-Civita symbols as a determinant of Kronecker deltas. We now see that equation (27) may be written

$$C^{\mu}\ C_{\mu} = R + \frac{1}{2}\ M^{\mu\nu i}\ M_{\mu\nu i} + 6 A^{\mu}\ A_{\mu} - 2 C^{\mu}_{;\mu} \tag{32}$$

The term $M^{\mu\nu i}\ M_{\mu\nu i}$ is the electroweak Lagrangian, and the $A^{\mu}\ A_{\mu}$ term has precisely the form that is needed (see, e.g., Moriyasu, 1983) for the introduction of mass.

5. FIELD EQUATIONS

We have previously (Pandres, 1981) considered the variational principle $\delta \int C^{\mu}\ C_{\mu}\ \sqrt{-g}\ d^4 x = 0$ where h^i_{μ} is varied. We note that $\sqrt{-g}$ equals h, the determinant of h^i_{μ}, and that $C^{\mu}\ C_{\mu} = C^i\ C_i$. Hence, our variational principle may be written

$$\delta \int C^i\ C_i\ h\ d^4\ x = 0 \tag{33}$$

The variational calculation (Pandres, 1984a) using $C^i C_i$ is less tedious than that using $C^{\mu} C_{\mu}$. We find from equation (33) that

$$\int h (2 C^i \delta C_i - C^i C_i h^k_{\nu} \delta\ h^{\nu}_k)\ d^4 x = 0 \tag{34}$$

where we have used $\delta h = h h^{\nu}_k\ \delta h^k_{\nu} = -h h^k_{\nu}\ \delta h^{\nu}_k$. We note that

$$\begin{aligned}(h h^{\nu}_i)_{,\nu} &= h_{,\nu}\ h^{\nu}_i + h h^{\nu}_{i,\nu} = h\ (h^{\mu}_k\ h^k_{\mu,\nu}\ h^{\nu}_i + h^{\nu}_{k,\nu}\ h^k_{\mu}\ h^{\mu}_i) \\ &= h\ (h^{\nu}_k\ h^k_{\nu,\mu}\ h^{\mu}_i - h^{\nu}_k\ h^k_{\mu,\nu}\ h^{\mu}_i) = -h C_{\mu}\ h^{\mu}_i = -h C_i\end{aligned}$$

Thus, we see that

$$C_i = -\ h^{-1}\ (h h^{\nu}_i)_{,\nu} \tag{35}$$

Variation of equation (35) gives

$$\delta C_i = h^{-2}\,(hh_i^{\nu})_{,\nu}\,\delta h - h^{-1}\,\delta\,(hh_i^{\nu})_{,\nu} = C_i\,h_\nu^k\,\delta h_k^{\nu} - h^{-1}\,[\delta\,(hh_i^{\nu})]_{,\nu}$$

Upon using this expression for δC_i in equation (34), we obtain

$$\int hC^k\,C_k h_\nu^i\,\delta h_i^{\nu}\,d^4x - 2\int C^i[\delta(hh_i^{\nu})_{,\nu}\,d^4x = 0 \tag{36}$$

and integration by parts gives

$$\int h(C^i_{,\nu} - h_\nu^i\,C^k_{,k} + \frac{1}{2}\,h_\nu^i C^k C_k)\,\delta h_i^{\nu}\,d^4x - \int [C^i\delta(hh_i^{\nu})]_{,\nu}\,d^4x = 0 \tag{37}$$

By using Gauss' theorem, we may write the second integral of equation (37) as an integral over the boundary of the region of integration. We discard this boundary integral by demanding that $C^i\delta(hh_i^{\nu})$ shall vanish on the boundary, and demand that δh_i^{ν} be arbitrary in the interior of the (arbitrary) region of integration. We get field equations $C^i_{,\nu} - h_\nu^i\,C^k_{,k} + 1/2\;h_\nu^i\,C^k\,C_k = 0$, and, upon multiplying by h_j^{ν}, we write these field equations as

$$C^i_{,j} - \delta_j^i\,C^k_{,k} + \frac{1}{2}\,\delta_j^i\,C^k C_k = 0 \tag{38}$$

5.1. The Field Equations as Einstein Equation

We note that

$$C^{\alpha}_{;\sigma} = (C^k h_k^{\alpha})_{;\sigma} = C^k_{,\sigma} h_k^{\alpha} + C^k h^{\alpha}_{k;\sigma} = C^k_{,\sigma}\,h_k^{\alpha} + C^k\gamma^{\alpha}_{k\sigma}$$

Thus, we have $C^k_{;\sigma}\,h_k^{\alpha} = C^{\alpha}_{;\sigma} + C^{\rho}\gamma^{\alpha}{}_{\rho\sigma}$. If we multiply by $h_\alpha^i\,h_j^{\sigma}$, we obtain $C^i_{,j} = h_\alpha^i\,h_j^{\sigma}\,(C^{\alpha}_{;\sigma} + C^{\rho}\gamma^{\alpha}{}_{\rho\sigma})$ and $C^k_{,k} = C^{\alpha}{}_{;\alpha} + C^{\alpha}\,C_{\alpha}$. Using these expressions for $C^i_{,j}$ and $C^k_{,k}$ in equation (38), we get the relation

$$h_\alpha^i\,h_j^{\sigma}\,(C^{\alpha}{}_{;\sigma} + C^{\rho}\gamma^{\alpha}{}_{\rho\sigma}) - \delta_j^i\,C^{\alpha}_{;\alpha} - \frac{1}{2}\,\delta_j^i\,C^{\alpha}\,C_{\alpha} = 0$$

and, upon multiplying this by $h_{i\mu}h_\nu^j$, we rewrite our field equations as

$$C_{\mu;\nu} - C_{\alpha}\gamma^{\alpha}{}_{\mu\nu} - g_{\mu\nu}\,C^{\alpha}_{;\alpha} - \frac{1}{2}\,g_{\mu\nu}\,C^{\alpha}C_{\alpha} = 0 \tag{39}$$

From equations (26) and (27), we find that an identity for the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - 1/2\ g_{\mu\nu}\ R$ is

$$G_{\mu\nu} = C_{\mu;\nu} - C_{\alpha}\gamma^{\alpha}{}_{\mu\nu} - g_{\mu\nu}\, C^{\alpha}_{;\alpha} - \frac{1}{2}\, g_{\mu\nu}\, C^{\alpha}\, C_{\alpha} \tag{40}$$

$$+ \gamma_{\mu}{}^{\alpha}{}_{\nu;\alpha} + \gamma^{\alpha}{}_{\sigma\nu}\gamma^{\sigma}{}_{\mu\alpha} + \frac{1}{2}\, g_{\mu\nu}\, \gamma^{\alpha i \sigma}\gamma_{\alpha\sigma i}$$

Equation (39) just states that the first line on the right side of equation (40) vanishes. Thus, we may write our field equations as

$$G_{\mu\nu} = \gamma_{\mu}{}^{\alpha}{}_{\nu;\alpha} + \gamma^{\alpha}{}_{\sigma\nu}\gamma^{\sigma}{}_{\mu\alpha} + \frac{1}{2}\, g_{\mu\nu}\, \gamma^{\alpha i \sigma}\, \gamma_{\alpha\sigma i} \tag{41}$$

We now show that the set of tetrads h^{i}_{α} which satisfy these field equations contains a nondenumerably infinite proper subset for which the condition $[\partial_{\mu}, \partial_{\nu}]h^{i}_{\alpha} = 0$ is satisfied, i.e., for which h^{i}_{α} is path-independent. Only after we have shown this can we write equation (41) in a form that makes its physical interpretation more evident.

5.2. Path-Independent Solutions of the Field Equations

5.2.1. First Integral of the Field Equations

Suppose that the tetrad h^{i}_{μ} satisfies our field equations. For distinct values of i and j, equation (38) becomes $C^{i}_{,j} = 0$. Thus, we see that the component C^{i} can depend only on the single coordinate x^{i}. The trace of equation (38) is $3C^{k}_{,k} - 2C^{k}C_{k} = 0$. Upon using this to eliminate $C^{k}_{,k}$ from equation (38), we obtain

$$C^{i}_{,j} = \frac{1}{6}\, \delta^{i}_{j} C^{k} C_{k} \tag{42}$$

If we set i and j equal to the same value N (no summation on N), we get $C^{N}_{,N} = 1/6\ C^{k}C_{k}$. It follows from this that $C^{0}_{,0} = C^{1}_{,1} = C^{2}_{,2} = C^{3}_{,3}$. But $C^{0}_{,0}$ can depend only on x^{0}; $C^{1}_{,1}$ only on x^{1}; etc. Thus, it is clear that $C^{N}_{,N}$ is a constant (same constant for all N); hence $C^{k}C_{k}$ is a constant. The constancy of $C^{k}C_{k}$ allows us to integrate equation (42). This integration gives $C^{i} = 1/6\ C^{k}C_{k}x^{i} + B^{i}$, where B^{i} = const. By using this expression for C^{i}, we obtain $C^{k}C_{k} = g_{ij}\ (1/6\ C^{m}C_{m}x^{i} + B^{i})\ (1/6\ C^{k}C_{k}x^{j} + B^{j})$. If we differentiate with respect to x^{n}, we obtain $C^{m}C_{m}\ (C^{k}C_{k}x^{n} + 6B^{n}) = 0$. Now, if $C^{k}C_{k}x^{n} + 6B^{n} = 0$, the constancy of $C^{k}C_{k}$ and B^{n} implies that $C^{k}C_{k}$ vanishes. Thus, C_{i} must either vanish or be lightlike. In either case, we see from equation (42) that $C^{i}_{,j} = 0$; hence, C^{i} must be constant. Our conclusion

(Pandres, 1984a) is that *a tetrad satisfies our field equations if and only if it yields a curvature vector whose Latin components C_i vanish or are constant and lightlike.*

5.2.2. Path-Independent Tetrads Which Yield $C_i = 0$

Consider the tetrad $h^i_\mu = \delta^i_\mu + \delta^i_0 \delta^2_\mu x^1$, where x^1 is a Greek (space-time) coordinate. We have shown (Pandres, 1981) that this tetrad yields $C_i = 0$ and gives a Ricci scalar $R = 1/2$. By contrast, the tetrad $h^i_\mu = \delta^i_\mu$ yields $\tilde{C}_i = 0$, but gives a Ricci scalar that vanishes. Thus, it is clear that two path-independent tetrads which satisfy our field equations and yield curvature vectors with identical Latin components are not generally related by a diffeomorphism.

We now recall (Pandres, 1995) that the set of tetrads which yield $C_i = 0$ contains a nondenumerably infinite subset of path-independent tetrads; i.e., tetrads which satisfy $[\partial_\mu, \partial_\nu] h^i_\alpha = 0$. Let $H^{\mu\nu}_i$ be four antisymmetric tensor densities of weight $+1$. The only conditions on the $H^{\mu\nu}_i$ are as follows:

1. They are path-independent functions, i.e., $H^{\mu\nu}_i = H^{\mu\nu}_i(x)$.
2. The vector densities of weight $+1$ defined by $H^\mu_i = H^{\mu\nu}_{i,\nu}$ are linearly independent.

From Condition 1, it follows that $[\partial_\alpha, \partial_\beta] H^{\mu\nu}_i = 0$. From Condition 2, it follows that H, the determinant of H^μ_i, is nonzero. This determinant is $H = 1/4!\ e_{\alpha\sigma\mu\nu}\ H^\alpha_i\ H^\sigma_j\ H^\mu_m\ H^\nu_n\ e^{ijmn}$. Since $e_{\alpha\sigma\mu\nu}$ is a tensor density of weight -1, it is clear that H is a scalar density of weight $+3$. Thus, $H^{-1/3}$ is a scalar density of weight -1, so that $H^{-1/3} H^\mu_i$ is a vector, i.e., it has weight zero. We define h^μ_i by

$$h^\mu_i = H^{-1/3}\ H^\mu_i \tag{43}$$

We note that $h = \text{Det}\ h^i_\mu = (\text{Det}\ h^\mu_i)^{-1} = \{\text{Det}\ [H^{-1/3} H^\mu_i]\}^{-1} = [H^{-4/3} H]^{-1} = H^{1/3}$. From this and equation (43), we see that $h h^\mu_i = H^\mu_i$. Thus, we find that $(h\ h^\mu_i)_{,\mu} = H^\mu_{i,\mu} = H^{\mu\nu}_{i,\nu,\mu} = 1/2\ (H^{\mu\nu}_{i,\nu,\mu} + H^{\mu\nu}_{i,\nu,\mu}) = 1/2\ (H^{\mu\nu}_{i,\nu,\mu} + H^{\nu\mu}_{i,\mu,\nu}) = 1/2\ (H^{\mu\nu}_{i,\nu,\mu} - H^{\mu\nu}_{i,\mu,\nu}) = 1/2\ [\partial_\mu, \partial_\nu]\ H^{\mu\nu}_i = 0$. Since $(h h^\mu_i)_{,\mu} = 0$, we see from equation (35) that $C_i = 0$.

5.2.3. Path-Independent Tetrads Which Yield C_i Constant and Lightlike

Consider the tetrad $h^i_\mu = \delta^i_\mu + (\delta^i_0 + \delta^i_1)\ \delta^0_\mu\ (e^{x^1} - 1)$, where the coordinate x^1 is Greek. We have shown (Pandres, 1984a) that this tetrad yields a nonvanishing but constant and lightlike C_i.

We have not yet found an explicit expression, like equation (43), for a nondenumerably infinite set of path-independent tetrads which yield a given constant lightlike value of C_i. It is clear, however, that such a set exists. Suppose, for example, that $C_i = \delta_i^0 + \delta_i^1$. Then, h^i_μ must satisfy the differential equations $h^{-1}\,(hh_i^\mu)_{,\mu} = -\,(\delta_i^0 + \delta_i^1)$. These equations comprise only four conditions on the 16 fields h^i_μ.

5.2.4. Nontrivial Solutions That Yield Flat Riemann Space-times

We note that our field equations admit nontrivial solutions for which $g_{\mu\nu}$ is the metric of a flat space-time. Green (1991) exhibited the tetrad

$$h_i^\mu = \delta_0^\mu\delta_i^0 + \delta_3^\mu\delta_i^3 + (\delta_1^\mu\delta_i^1 + \delta_2^\mu\delta_i^2)\cos x^3 + (\delta_2^\mu\delta_i^1 - \delta_1^\mu\delta_i^2)\sin x^3 \tag{44}$$

where the coordinate x^3 is Greek. For this h_i^μ, the quantity $M_{\mu\nu i}$ does not vanish, but $C_i = 0$, and $g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. He also exhibited (Green, 1997) the tetrad

$$h_i^\mu = \frac{1}{2}\left[(\delta_0^\mu\,\delta_i^0 + \delta_1^\mu\,\delta_i^1)\left(F + \frac{1}{F}\right)\right] + \frac{1}{2}\left[(\delta_0^\mu\delta_i^1 + \delta_1^\mu\delta_i^0)\left(F - \frac{1}{F}\right)\right] + \delta_2^\mu\,\delta_i^2 + \delta_3^\mu\,\delta_i^3 \tag{45}$$

where $F = x^0 + x^1$, and the coordinates x^0 and x^1 are Greek. For this h_i^μ, the quantity $M_{\mu\nu i}$ does not vanish, but C_i is constant and lightlike, and $g_{\mu\nu} = \mathrm{diag}\,(-1, 1, 1, 1)$.

5.3. Einstein Equations Revisited

In Section 5.2 we saw seen that the set of tetrads h^i_α which satisfy our field equations contains a nondenumerably infinite proper subset of path-independent tetrads. Any h^i_α in this subset appears in the guise of a tetrad defined on a manifold. In the remainder of this paper, we limit our considerations to such path-independent tetrads. Since h^i_μ is path independent, the metric $g_{\mu\nu} = g_{ij}h^i_\mu h^j_\nu$ and all of its derivatives are path independent.

We may now use the well-known symmetry of the Einstein tensor, i.e., $G_{\mu\nu} = G_{\nu\mu}$. (In the usual proof of this symmetry, it is assumed that $g_{\mu\nu}$ is path independent.) Thus, we see from equation (41) that the symmetric part of our field equations is

$$G_{\mu\nu} = \frac{1}{2}\,(\gamma_\mu{}^\alpha{}_\nu + \gamma\nu^\alpha{}_\mu)_{;\alpha} + \frac{1}{2}\,(\gamma^\alpha{}_{\sigma\nu}\gamma^\sigma{}_{\mu\alpha} + \gamma^\alpha{}_{\sigma\mu}\gamma^\sigma{}_{\nu\alpha}) + \frac{1}{2}\,g_{\mu\nu}\gamma^{\alpha i\sigma}\gamma_{\alpha\sigma i} \tag{46}$$

Since $\gamma_\mu{}^\alpha{}_\nu = M_\mu{}^\alpha{}_\nu + A_\mu{}^\alpha{}_\nu$, we see that

$$(\gamma_\mu{}^\alpha{}_\nu + \gamma_\nu{}^\alpha{}_\mu)_{;)\alpha} = (M_\mu{}^\alpha{}_\nu + M_\nu{}^\alpha{}_\mu)_{;\alpha} = (M_\mu{}^\alpha{}_i h^i{}_\nu + M_\nu{}^\alpha{}_i h^i{}_\mu)_{;\alpha}$$
$$= J_{\mu i} h^i{}_\nu + J_{\nu i} h^i{}_\mu + M_\mu{}^\alpha{}_\sigma \gamma^\sigma{}_{\nu\alpha} + M_\nu{}^\alpha{}_\sigma \gamma^\sigma{}_{\mu\alpha}$$

where $J_{\mu i} = M_\mu{}^\alpha{}_{i;\alpha}$ is a (conserved) electroweak current. From equation (28), the repeated use of equation (29), the total antisymmetry of $A_{\mu\nu\alpha}$, and the antisymmetries of $\gamma_{\mu\nu\alpha}$ and $M_{\mu\nu\alpha}$ in their first two indices, we find after a tedious but straightforward calculation that equation (46) may be written

$$G_{\mu\nu} = A^{ij}{}_\mu A_{ij\nu} - \frac{1}{2} g_{\mu\nu} A^{ij\alpha} A_{ij\alpha} + \frac{1}{2}(J_{\mu i} h^i_\nu + J_{\nu i} h^i_\mu) - M_{\mu\nu} \tag{47}$$

where $M_{\mu\nu} = M^\alpha{}_{\mu i} M_{\alpha\nu}{}^i - \frac{1}{4} g_{\mu\nu} M^{\alpha\sigma i} M_{\alpha\sigma i}$. The terms in equation (47) that involve $A_{ij\mu}$ may be written in a simpler form. From equation (30), we have $A_\mu = (1/3!)(-g)^{-1/2} g_{\mu\rho} e^{\rho\alpha\beta\sigma} A_{\alpha\beta\sigma}$, and we find after a little work that $A_\mu = -(1/3!)(-g)^{1/2} e_{\mu\alpha\beta\sigma} A^{\alpha\beta\sigma}$. Thus, $A_\mu A_\nu = -\frac{1}{36} g_{\mu\rho} e^{\rho\alpha\beta\sigma} e_{\nu\theta\lambda\tau} A^{\theta\lambda\tau} A_{\alpha\beta\sigma}$. By expressing the product of Levi-Civita symbols as a determinant of Kronecker deltas, we get $A_\mu A_\nu = \frac{1}{2} A^{ij}{}_\mu A_{ij\nu} - \frac{1}{6} g_{\mu\nu} A^{ij\alpha} A_{ij\alpha}$. From this and equation (31), we see that equation (47) may be written

$$G_{\mu\nu} = 2A_\mu A_\nu + g_{\mu\nu} A^\alpha A_\alpha + \frac{1}{2}(J_{\mu i} h^i_\nu + J_{\nu i} h^i_\mu) - M_{\mu\nu} \tag{48}$$

The right side of equation (48) is just what one would expect for the stress-energy tensor of the electroweak field, its associated currents, and gauge symmetry-breaking terms corresponding to those in the Lagrangian of equation (32).

6. AN ALTERNATIVE THEORY

Schrödinger (1960) recognized that the simplest general relativistic variational principle which exists is

$$\delta \int \sqrt{-g}\, d^4x = 0 \tag{49}$$

where g is the determinant of the metric $g_{\mu\nu}$. He noted, however, that variation of $g_{\mu\nu}$ yields the Euler–Lagrange equations $\sqrt{-g} g^{\mu\nu} = 0$, which cannot serve as field equations. If one expresses $g_{\mu\nu}$ in terms of the tetrad and varies the

16 components of h^i_μ independently, one gets the Euler–Lagrange equations $\sqrt{-g}h^\mu_i = 0$, which also cannot serve as field equations. We have considered (Pandres, 1995, 1998) a theory in which the vectors of the tetrad h^i_μ are expressed as derivatives of "nonintegrable functions" x^i and the four x^i are varied independently. [A nonintegrable function does not have a definite numerical value at a point, but its derivatives have definite values at a point. Such nonintegrable functions (path-dependent functions with path-independent derivatives) have been used as phase factors by Dirac (1978), Yang (1974), and many others in gauge theory.] Any path-independent tetrad may be expressed in this way; thus, without loss of generality, we may write $h^i_\mu = x^i_{,\mu}$, where a comma denotes partial differentiation. Since $\sqrt{-g}$ equals $\mathfrak{J}$, the (Jacobian) determinant of $x^i_{,\mu}$, our variational principle may be written

$$\delta \int \mathfrak{J}\, d^4x = 0 \tag{50}$$

Clearly, this may be viewed as a "principle of stationary space-time volume."

6.1. Field Equations

By using the formula for the derivative of a determinant, we have $\delta\mathfrak{J} = \mathfrak{J}\, h^\mu_i\, \delta h^i_\mu$, and since $\delta h^i_\mu = \delta(x^i_{,\mu}) = (\delta x^i)_{,\mu}$, equation (50) may be written $\int \mathfrak{J}\, h^\mu_i\, (\delta x^i)_{,\mu}\, d^4x = 0$. Upon integrating by parts, we have $\int(\mathfrak{J}\, h^\mu_i\, \delta x^i)_{,\mu}\, d^4x - \int(\mathfrak{J}\, h^\mu_i)_{,\mu}\, \delta x^i\, d^4x = 0$. By using Gauss' theorem, we write $\int(\mathfrak{J}\, h^\mu_i\, \delta x^i)_{,\mu}\, d^4x$ as a boundary integral which we discard by demanding that δx^i vanish on the boundary. Thus, we obtain $\int(\mathfrak{J}\, h^\mu_i)_{,\mu}\, \delta x^i\, d^4x = 0$, and by demanding that δx^i be arbitrary in the interior of the region of integration, we obtain the field equations $(\mathfrak{J}\, h^\mu_i)_{,\mu} = 0$. Now,

$$\begin{aligned}
(\mathfrak{J}h^\alpha_j)_{,\alpha} &= \mathfrak{J}\, h^\alpha_{j,\alpha} + h^\alpha_j \mathfrak{J}_{,\alpha} \\
&= \mathfrak{J}h^\alpha_{j,\nu}\delta^\nu_\alpha + h^\alpha_j \mathfrak{J} h^\nu_i h^i_{\nu,\alpha} \\
&= \mathfrak{J}h^{\ \alpha}_{j,\nu}\, h^i_\alpha\, h^\nu_i + \mathfrak{J}h^\alpha_j\, h^\nu_i\, h^i_{\nu,\alpha} \\
&= -\mathfrak{J}h^\alpha_j\, h^i_{\alpha,\nu} h^\nu_i + \mathfrak{J}h^\alpha_j h^\nu_i h^i_{\nu,\alpha} \\
&= \mathfrak{J}h^\alpha_j h^\nu_i\, (h^i_{\nu,\alpha} - h^i_{\alpha,\nu})
\end{aligned}$$

Upon multiplying this by h^j_μ, we see that our field equations may be written $C_\mu = 0$, where C_μ is the curvature vector defined in equation (23). Thus, we see that any tetrad which satisfies the field equations of our alternative theory also satisfies the field equations of the theory developed in Sections 1–5; however, the converse is not true.

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